# A Folk Theorem for Discounted Repeated Games

Game Theory Course:
Jackson, Leyton-Brown & Shoham

# A (Simple) Folk Theorem for Discounted Repeated Games

- Consider a finite normal form game $G = (N, A, u)$.

- Let $a = (a_1, \ldots, a_n)$ be a Nash equilibrium of the stage game $G$

  If $a = (a_1, \ldots, a'_n)$ is such that $u_i(a) > u_i(a)$ for all $i$, then there exists a discount factor $\beta < 1$, such that if $\beta_i \geq \beta$ for all $i$, then there exists a subgame perfect equilibrium of the infinite repetition of $G$ that has $a'$ played in every period on the equilibrium path.

# A (Simple) Folk Theorem for Discounted Repeated Games

- Outline of the Proof:

- Play $a'$ as long as everyone has in the past.

- If any player ever deviates, then play $a$ forever after (Grim Trigger).

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  - Deviation is not beneficial if $\frac{M}{m} \leq \frac{\beta_i}{1-\beta_i}$ or $\beta_i \geq \frac{M}{M+m}$ for all $i$.

# Repeated Prisoner's Dilemma

- More complicated play: something to think about

| | C | D |
|---|---|---|
| C | 3,3 | 0,10 |
| D | 10,0 | 1,1 |

# Repeated Games

- Players can condition future play on past actions
- Brings in many(!) equilibria: Folk Theorems
- Need key ingredients
  - Some (fast enough) observation about how others behave
  - Sufficient value to the future (limit of the means - extreme value) or high enough discount factor